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Symmetries and a hierarchy of the general modified KdV equation

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Abstract. Two groups of symmetries and their Lie algebra properties for the modified KdV equation are extended to the general modified KdV equation and the Miura transformation between the general KdV equation and the general modified KdV equation is also established.

In Tian Chou (1985, 1987), we discussed the general KdV (GKdV) equation

$$u_t + u_{xxx} + 6uu_x + 6fu - x(f' + 12f^2) = 0 \tag{1}$$

where f is an arbitrary function of t. For the GKdV equation, we have found its Lax pair:

$$\Omega = \begin{pmatrix} 0 & 1 \\ k-u & 0 \end{pmatrix} \mathrm{d}x + \begin{pmatrix} (u+2k)_x & -(4k+2u) \\ u_{xx} - (4k+2u)(k-u) & -(u+2k)_x \end{pmatrix} \mathrm{d}t \tag{2}$$

where $k = xf + \lambda g$, λ is an arbitrary constant and $g(t) = \exp(-\int 12f\,\mathrm{d}t)$ $(g' + 12fg = 0)$. Writing the Lax equation

$$\mathrm{d}a = \Omega a \qquad a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

as the Riccati form, we have

$$\varphi_x = k - u - \varphi^2 \tag{3}$$

$$\varphi_t = -(4k+2u)(k - u - \varphi^2) + u_{xx} - 2(u_x + 2f)\varphi \tag{4}$$

$(\varphi = a_2/a_1)$.

From (3)

$$u = k - \varphi_x - \varphi^2.$$

Substituting this into (4), we obtain

$$\varphi_t + \varphi_{xxx} - 6\varphi^2\varphi_x + 6k\varphi_x + 6f\varphi = 0. \tag{5}$$

Therefore we have the transformation between the GKdV equation (1) and equation (5):

$$u = xf + \lambda g - \varphi_x - \varphi^2. \tag{6}$$

In particular, when $\lambda = 0$, equation (5) is reduced to

$$\varphi_t + \varphi_{xxx} - 6\varphi^2\varphi_x + 6xf\varphi_x + 6f\varphi = 0 \tag{7}$$

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and (6) is reduced to

$$u = xf - \varphi_x - \varphi^2. \tag{8}$$

If we change φ to $i\tilde{\varphi}$, then (7) is changed to

$$\tilde{\varphi}_t + \tilde{\varphi}_{xxx} + 6\tilde{\varphi}^2\tilde{\varphi}_x + 6xf\tilde{\varphi}_x + 6f\tilde{\varphi} = 0. \tag{9}$$

When $f = 0$, (1) is reduced to the KdV equation

$$u_t + u_{xxx} + 6uu_x = 0$$

and (7) or (9) is reduced to the modified KdV equation

$$\varphi_t + \varphi_{xxx} - 6\varphi^2\varphi_x = 0$$

or

$$\tilde{\varphi}_t + \tilde{\varphi}_{xxx} + 6\tilde{\varphi}^2\tilde{\varphi}_x = 0$$

and transformation (6) is reduced to the Miura transformation

$$u = -\varphi_x - \varphi^2.$$

We call (7) or (9) the general modified KdV (GMKdV) equation and (8) is the Miura transformation as well.

Since the GMKdV equation is invariant when we change φ to $-\varphi$, the transformation (8) can be written as

$$u = xf + \varphi_x - \varphi^2. \tag{10}$$

We can establish the Miura transformation, (10) or (8), directly. In fact, substituting (10) into (1), we can obtain

$$(\mathrm{D} - 2\varphi)(\varphi_t + \varphi_{xxx} - 6\varphi^2\varphi_x + 6xf\varphi_x + 6f\varphi) = 0$$

($\mathrm{D} = \mathrm{d}/\mathrm{d}x$).

It is trivial that φ is a solution of (5) if and only if $-\varphi$ is a solution of (5) as well. Therefore, we can change equations (3) and (4) to

$$-\varphi_x = k - u' - \varphi^2 \tag{11}$$

$$-\varphi_t = -(4k + 2u')(k - u' - \varphi^2) + u'_{xx} + 2(u'_x + 2f)\varphi. \tag{12}$$

Then u' is also a solution of (1) if φ satisfies (3) and (4). From (3) and (4), we obtain the relation between u and u':

$$u' = u + 2\varphi_x.$$

This is a Bäcklund transformation (in Darboux type) of the GKdV equation and is the same as the result in Tian Chou (1987).

The above discussion can be considered as an extension of the results in Chen (1974).

Furthermore, we discuss the symmetries of the GMKdV equation

$$\begin{aligned} \varphi_t &= K \\ K &= -(\varphi_{xxx} - 6\varphi^2\varphi_x + 6xf\varphi_x + 6f\varphi). \end{aligned} \tag{13}$$

At first, we can check that

$$\Phi = (1/g)(\mathrm{D}^2 + 4\varphi^2 + 4\varphi_x\mathrm{D}^{-1}\varphi) \tag{14}$$

(D^{-1} is the inverse operator of D) satisfies the equation

$$\mathrm{d}\Phi/\mathrm{d}t = [K', \Phi]$$

where $K' = -[D^3 + (xf - \varphi^2)D - 12\varphi\varphi_x + 6f]$ (see the appendix). Hence, Φ is a strong symmetry (or recursion operator) of the GMKdV equation (Oevel and Fokas 1984, Tian Chou 1985). It is easy to show that Φ is a hereditary symmetry (Olver 1977). Therefore, a hierarchy of the GMKdV equation is generated by Φ and (13):

$$u_t = K_n \qquad K_n = \Phi^n K \qquad n = 0, 1, 2, \ldots \tag{15}$$

and Φ is the strong symmetry of all of equations (15).

We point out that there is a transformation which links the modified KdV equation $\psi_\tau + \psi_{\xi\xi\xi} - 6\psi^2\psi_\xi = 0$ to the GMKdV equation:

$$\varphi_t + \varphi_{xxx} 6\varphi^2\varphi_x + 6xf\varphi_x + 6f\varphi = 0$$

$$\varphi = \sqrt{g}\,\psi \qquad \xi = g^{1/2}x \qquad \tau = \int g^{3/2}\,\mathrm{d}t.$$

However, this transformation cannot transform the MKdV equations of high order into the GMKdV equations of high order.

Next, we present two symmetries of the GMKdV equation as follows:

$$\sigma_0 = (1/\sqrt{g})\varphi_x$$

$$\tau_1 = 3g^{-3/2} \int g^{3/2}\,\mathrm{d}t(\varphi_t + 6f(x\varphi)_x) + (x\varphi)_x$$

$$= 3\int g^{3/2}\,\mathrm{d}t\,\Phi\sigma_0 + (x\varphi)_x.$$

(We can check that σ_0 and τ_1 satisfy the equations $\mathrm{d}\sigma_0/\mathrm{d}t = K'[\sigma_0]$ and $\mathrm{d}\tau_1/\mathrm{d}t = K'[\tau_1]$ directly.) Therefore, two groups of symmetries are generated by σ_0, τ_1 and Φ:

$$\sigma_m = \Phi^m\sigma_0 \qquad m = 0, 1, 2, \ldots$$

$$\tau_n = \Phi^{n-1}\tau_1$$

$$= 3\int g^{3/2}\,\mathrm{d}t\,\sigma_n + \Phi^{n-1}(x\varphi)_x \qquad n = 1, 2, \ldots.$$

Theorem. σ_m $(m = 0, 1, 2, \ldots)$ and τ_n $(n = 1, 2, \ldots)$ satisfy the Lie algebra

(i) $[\sigma_m, \sigma_n] = 0 \qquad m, n = 0, 1, 2, \ldots$

(ii) $[\sigma_m, \tau_n] = (2m+1)\sigma_{m+n-1} \qquad m = 0, 1, 2, \ldots,\ n = 1, 2, \ldots$

(iii) $[\tau_m, \tau_n] = 2(m-n)\tau_{m+n-1} \qquad m, n = 1, 2, \ldots$

($[a, b] = a'[b] - b'[a]$).

To prove this theorem, we need the following lemmas.

Lemma 1.

$$\Phi'[\sigma_m] = [\sigma'_m, \Phi] \qquad m = 0, 1, 2, \ldots. \tag{16}$$

Proof. First, we can check that

$$\Phi'[\sigma_0] = [\sigma'_0, \Phi]. \tag{17}$$

and (17) is equivalent to

$$\Phi[\sigma_0, a] = [\sigma_0, \Phi a]$$

for any function a, i.e. Φ commutes with σ_0 (Fuchssteiner 1981). Next, since Φ is a hereditary symmetry, we can also prove that Φ commutes with σ_n ($n = 1, 2, \ldots$) (Tian Chou 1987). Therefore, (16) is valid.

Lemma 2.

$$[\sigma_0, (x\varphi)_x] = \sigma_0.$$

Proof.

$$\begin{aligned}[\sigma_0, (x\varphi)_x] &= \sigma_0'[(x\varphi)_x] - ((x\varphi)_x)'[\sigma_0] \\ &= (1/\sqrt{g})(x\varphi)_{xx} - x(\sigma_0)_x - \sigma_0 \\ &= \sigma_0.\end{aligned}$$

Lemma 3.

$$\Phi'[(x\varphi)_x] + \Phi \circ ((x\varphi)_x)' - ((x\varphi)_x)' \circ \Phi = 2\Phi. \tag{18}$$

Proof. Notice that $D^{-1}(x\varphi D) = x\varphi - D^{-1}((x\varphi)_x) - D^{-1}\varphi$, we can check it directly.

Lemma 4.

$$[\sigma_m, (x\varphi)_x] = (2m+1)\sigma_m. \tag{19}$$

Proof. According to lemma 2, (19) is established for $n = 0$. Notice that

$$(\Phi a)'[b] = \Phi'[b]a + \Phi a'[b] \tag{20}$$

is valid for any operator Φ and any functions a and b (Li and Zhu 1985a, b), then we can prove this lemma by using induction and lemma 3.

Lemma 5.

$$[\sigma_m, \Phi^{n-1}(x\varphi)_x] = (2m+1)\sigma_{m+n-1} \qquad m = 0, 1, 2, \ldots, n = 1, 2, \ldots. \tag{21}$$

Proof. According to lemma 4, (21) is valid for any m when $n = 1$. Suppose (21) is established for $n = k$, i.e.

$$[\sigma_m, \Phi^{k-1}(x\varphi)_x] = (2m+1)\sigma_{m+k-1}.$$

Using (20) and lemma 1

$$\begin{aligned}[\sigma_m, \Phi^k(x\varphi)_x] &= \sigma_m'[\Phi^k(x\varphi)_x] - (\Phi^k(x\varphi)_x)'[\sigma_m] \\ &= \sigma_m'[\Phi^k(x\varphi)_x] - \Phi'[\sigma_m]\Phi^{k-1}((x\varphi)_x) - \Phi(\Phi^{k-1}(x\varphi)_x)'[\sigma_m] \\ &= \sigma_m'[\Phi^k(x\varphi)_x] - [\sigma_m', \Phi]\Phi^{k-1}((x\varphi)_x) - \Phi(\Phi^{k-1}(x\varphi)_x)'[\sigma_m] \\ &= \Phi[\sigma_m, \Phi^{k-1}(x\varphi)_x] \\ &= (2m+1)\sigma_{m+k}.\end{aligned}$$

Therefore, (21) is valid.

Lemma 6.

$$[\Phi(x\varphi)_x, (x\varphi)_x] = 2\Phi(x\varphi)_x.$$

Proof. Using (18), we can check it directly.

Lemma 7.

$$[\Phi^{m-1}(x\varphi)_x, (x\varphi)_x] = 2m\Phi^{m-1}(x\varphi)_x \qquad m = 2, 3, \ldots.$$

Proof. According to lemma 6, this is valid when $m = 2$. Then we can prove it by using (18), (20) and induction.

Lemma 8.

$$[\Phi^{m-1}(x\varphi)_x, \Phi^{n-1}(x\varphi)_x] = 2(m-n)\Phi^{m+n-2}(x\varphi)_x \qquad m, n = 1, 2, \ldots. \tag{22}$$

Proof. According to lemma 7, (22) is valid for any m when $n = 1$. Using (20), (18) and the hereditary property of Φ (Fuchssteiner 1981), we can prove

$$\Phi'[\Phi^{n-1}(x\varphi)_x]\Phi^{m-1}(x\varphi)_x - (\Phi^{n-1}(x\varphi)_x)'[\Phi(x\varphi)_x]$$
$$= \Phi^{n-1}\Phi'[(x\varphi)_x]\Phi^{m-1}(x\varphi)_x - \Phi(\Phi^{n-1}(x\varphi)_x)'[\Phi(x\varphi)_x].$$

Then we can prove (22) by using induction.

Proof of the theorem.

(i) is the direct result of lemma 1.

(ii) From lemma 5

$$\begin{aligned}[\sigma_m, \tau_n] &= [\sigma_m, 3h\sigma_n + \Phi^{n-1}(x\varphi)_x] \qquad \left(h = \int g^{3/2}\,\mathrm{d}t\right)\\ &= [\sigma_m, \Phi^{n-1}(x\varphi)_x]\\ &= (2m+1)\sigma_{m+n-1}.\end{aligned}$$

(iii) From lemmas 5 and 8

$$\begin{aligned}[\tau_m, \tau_n] &= [3h\sigma_m + \Phi^{m-1}(x\varphi)_x, 3h\sigma_n + \Phi^{n-1}(x\varphi)_x]\\ &= 3h[\sigma_m, \Phi^{n-1}(x\varphi)_x] + 3h[\Phi^{m-1}(x\varphi)_x, \sigma_n] + [\Phi^{m-1}(x\varphi)_x, \Phi^{n-1}(x\varphi)_x]\\ &= 6h(m-n)\sigma_{m+n-1} + 2(m-n)\Phi^{m+n-2}(x\varphi)_x\\ &= 2(m-n)\tau_{m+n-1}.\end{aligned}$$

We complete the proof.

For the GMKdV equation

$$\varphi_t + \varphi_{xxx} \pm 6\varphi^2\varphi_x + 6xf\varphi_x + 6f\varphi = 0$$

there is a Lax pair as follows:

$$\Omega = M\,\mathrm{d}x + N\,\mathrm{d}t \tag{23}$$

where

$$M = \begin{pmatrix} \eta g^{1/2} & \varphi \\ -\varphi & -\eta g^{1/2} \end{pmatrix}$$

$$N = \begin{pmatrix} -4\eta^3 g^{3/2} - 2\eta g^{1/2}(\pm\varphi^2 + 3xf) & -(\varphi_{xx} + 2\varphi(2\eta^2 g \pm \varphi^2 + 3xf) - 2\eta g^{1/2}\varphi_x) \\ \varphi_{xx} + 2\varphi(2\eta^2 g \pm \varphi^2 + 3xf) - 2\eta g^{1/2}\varphi_x & 4\eta^3 g^{3/2} + 2\eta g^{1/2}(\pm\varphi^2 + 3xf) \end{pmatrix}$$

(η is an arbitrary constant), i.e. $d\Omega - \Omega \wedge \Omega = 0$ if and only if $\varphi_t + \varphi_{xxx} \pm 6\varphi^2\varphi_x + 6xf\varphi_x + 6f\varphi = 0$. In particular, when $f = 0$, $g = 1$ and (23) is reduced to

$$\Omega = \begin{pmatrix} \eta & \varphi \\ -\varphi & -\eta \end{pmatrix} dx + \begin{pmatrix} -(4\eta^3 \pm 2\eta\varphi^2) & -(\varphi_{xx} + 4\eta^2\varphi \pm 2\varphi^3) - 2\eta\varphi_x \\ \varphi_{xx} + 4\eta^2\varphi \pm 2\varphi^3 - 2\eta\varphi_x & 4\eta^3 \pm 2\eta\varphi^2 \end{pmatrix} dt.$$

This is a well known Lax pair of the modified KdV equation. Using the Lax pair (23), we can obtain a Bäcklund transformation (in Darboux type) and some solutions of the GMKdV equation.

The above results can be extended to the equation

$$\varphi_t + \varphi_{xxx} \pm 6\varphi^2\varphi_x + 6xf\varphi_x + 6l\varphi_x + 6f\varphi = 0$$

where f and l are arbitrary functions of t. It corresponds to the general KdV equation

$$u_t + u_{xxx} + 6uu_x - x(f' + 12f^2) - (l' + 12lf) = 0.$$

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Appendix

We prove that

$$\Phi = (1/g)(D^2 - 4\varphi^2 - 4\varphi_x D^{-1}\varphi)$$

satisfies the equation

$$d\Phi/dt = [K', \Phi]$$

i.e.

$$\partial\Phi/\partial t + \Phi'[K] = [K', \Phi]$$

where

$$K' = -(D^3 - 6\varphi^2 D + 6xfD - 12\varphi\varphi_x + 6f).$$

First, we need the following formulae:

$$D^2(\varphi^2 D) = \varphi^2 D^3 + 4\varphi\varphi_x D^2 + 2\varphi_x^2 D + 2\varphi\varphi_{xx} D$$

$$D^2(\varphi\varphi_x) = \varphi\varphi_x D^2 + 2\varphi\varphi_{xx} D + 2\varphi_x^2 D + \varphi\varphi_{xxx} + 3\varphi_x\varphi_{xx}$$

$$D^2(xD) = xD^3 + 2D^2$$

$$D^3\varphi^2 = \varphi^2 D^3 + 6\varphi\varphi_x D^2 + 6(\varphi\varphi_{xx} + \varphi_x^2)D + 2\varphi\varphi_{xxx} + 6\varphi_x\varphi_{xx}$$

$$D(\varphi_x D^{-1}\varphi) = \varphi_{xx} D^{-1}\varphi + \varphi\varphi_x$$

$$D^2(\varphi_x D^{-1}\varphi) = \varphi_{xxx} D^{-1}\varphi + 2\varphi\varphi_{xx} + \varphi_x^2 + \varphi\varphi_x D$$

$$D^3(\varphi_x D^{-1}\varphi) = \varphi_{xxxx} D^{-1}\varphi + 3\varphi\varphi_{xxx} + 4\varphi_x\varphi_{xxx} + 3\varphi\varphi_{xx} D + 2\varphi_x^2 D + \varphi\varphi_x D^2$$

$$D^{-1}(\varphi D) = -D^{-1}\varphi_x + \varphi$$
$$D^{-1}(\varphi^3 D) = -3D^{-1}\varphi^2\varphi_x + \varphi^3$$
$$D^{-1}(\varphi D^3) = \varphi D^2 - \varphi_x D + \varphi_{xx} - D^{-1}\varphi_{xxx}$$
$$D^{-1}(x\varphi D) = x\varphi - D^{-1}x\varphi_x - D^{-1}\varphi.$$

Next, we have

$$g\,d\Phi/dt = 12f(D^2 - 4\varphi^2 - 4\varphi_x D^{-1}) - 8\varphi\varphi_t - 4\varphi_{xt}D^{-1}\varphi - 4\varphi_x D^{-1}\varphi_t$$

and

$$g[K', \Phi] = [K', D^2 - 4\varphi^2 - 4\varphi_x D^{-1}\varphi].$$

Since

$$\begin{aligned}-4K'\circ(\varphi_x D^{-1}\varphi) &= 4(D^3 - 6\varphi^2 D + 6xfD - 12\varphi\varphi_x + 6f)\circ(\varphi_x D^{-1}\varphi)\\ &= 4[D^3(\varphi_x D^{-1}\varphi) + 6(xf - \varphi^2)D(\varphi_x D^{-1}\varphi)\\ &\quad - 12\varphi\varphi_x^2 D^{-1}\varphi + 6f\varphi_x D^{-1}\varphi]\\ &= 4(-\varphi_{xt}D^{-1}\varphi + 3\varphi\varphi_{xxx} + 4\varphi_x\varphi_{xx} + 6(xf - \varphi^2)\varphi\varphi_x\\ &\quad + 3\varphi\varphi_{xx}D + 2\varphi_x^2 D + \varphi\varphi_x D^2)\end{aligned}$$

and

$$\begin{aligned}4\varphi_x D^{-1}\varphi\circ K' &= -4(\varphi_x D^{-1}\varphi)\circ(D^3 - 6\varphi^2 D + 6xfD - 12\varphi\varphi_x + 6f)\\ &= -4[\varphi_x D^{-1}\varphi_t + \varphi_x\varphi_{xx} + 6(xf - \varphi^2)\varphi\varphi_x - \varphi_x^2 D\\ &\quad + \varphi\varphi_x D^2 + 6f\varphi_x D^{-1}\varphi]\end{aligned}$$

we have

$$\begin{aligned}[K', -4\varphi_x D^{-1}\varphi] &= -4K^0\circ\varphi_x D^{-1}\varphi + 4\varphi_x D^{-1}\varphi\circ K'\\ &= -4(\varphi_x D^{-1}\varphi_t + \varphi_{xt}D^{-1}\varphi - 3\varphi\varphi_{xxx} - 3\varphi_x\varphi_{xx} - 3\varphi\varphi_x D - 3\varphi_x^2 D + 6f\varphi_x D^{-1}\varphi).\end{aligned}$$

Since

$$\begin{aligned}K'\circ(D^2 - 4\varphi^2) &= -(D^3 - 6\varphi^2 D + 6xfD - 12\varphi\varphi_x + 6f)\circ(D^2 - 4\varphi^2)\\ &= -(D^5 + 6(xf - \varphi^2)D^3 - 12\varphi\varphi_x D^2 + 6fD^2\\ &\quad - 4[\varphi^2 D^3 + 6\varphi\varphi_x D^2 + 6\varphi_x\varphi_{xx}D + 6\varphi_x^2 D + 2\varphi\varphi_{xx} + 6\varphi_x\varphi_{xx} - 24(xf - \varphi^2)\\ &\quad \times(\varphi^2 D + 2\varphi\varphi_x) + 48\varphi^3\varphi_x - 24f\varphi^2]\end{aligned}$$

and

$$\begin{aligned}-(D^2 - 4\varphi^2)\circ K' &= (D^2 - 4\varphi^2)\circ(D^3 - 6\varphi^2 D + 6xfD - 12\varphi\varphi_x + 6f)\\ &= D^5 + 6(xf - \varphi^2)D^3 - 24\varphi\varphi_x D^2 + 12fD^2 - 12\varphi\varphi_{xx}D - 12\varphi_x^2 D\\ &\quad - 12(\varphi\varphi_x D^2 + 2\varphi_x\varphi_{xx}D + 2\varphi_x^2 D + \varphi\varphi_{xxx} + 3\varphi_x\varphi_{xx})\\ &\quad + 6fD^2 - 4\varphi^2 D^3 - 24\varphi^2(xf - \varphi^2)D + 48\varphi^3\varphi_x - 24f\varphi^2\end{aligned}$$

we have

$$[K', D^2 - 4\varphi^2] = -4\varphi\varphi_{xxx} - 12\varphi_x\varphi_{xx} + 48(xf - \varphi^2)\varphi\varphi_x + 12fD^2 - 12\varphi\varphi_{xx}D - 12\varphi_x^2 D$$

and

$$[K', D^2 - 4\varphi^2 - 4\varphi_x D^{-1}\varphi] = [K', D^2 - 4\varphi^2] + [K', -4\varphi_x D^{-1}\varphi]$$

$$= -4\varphi_x D^{-1}\varphi_t - 4\varphi_{xt} D^{-1}\varphi + 6f\varphi_x D^{-1}\varphi + 8\varphi$$

$$\times(\varphi_{xxx} + 6(xf - \varphi^2)\varphi\varphi_x) + 12f D^2$$

$$= -4\varphi_x D^{-1}\varphi_t - 4\varphi_{xt} D^{-1}\varphi + 6f\varphi_x D^{-1}\varphi$$

$$+ 12f D^2 - 8\varphi\varphi_t + 48f\varphi^2.$$

Therefore

$$d\Phi/dt = [K', \Phi].$$

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